One single and unique time-constant for a linear second order dynamical system

Dr. Marcelin Nini DABO, PhD.

***Abstract*—Modern control engineering presents dynamical systems as systems with a well-defined time-constant for first order ordinary differential equations (ODEs) and as systems without a time-constant for second order ODEs. In this paper, I present, for the first time, a new vision on control systems: there is only one single and unique time-constant for a second order dynamical system, whatever its damping ratio and natural pulsation, thanks to a deep observation and contemplation of solutions of second order ODEs. Thanks to this new approach, one can compute, with precision, temporal (or frequencial) performances for any given stable second order dynamical system. Also, it is absolutely possible to design a second order dynamical system with precision, by choosing the type of desired dynamics (critically damped, overdamped, underdamped) and the desired time-response or rise-time performance. Morever, a very simple criteria of identification for second order ODEs is presented.**

***Index Terms*—Ordinary Differential Equations (ODEs), second-order, time-constant, surface time-constant, volume time-constant, n-dimensional time-constant, circular trigonometry, hyperbolic trigonometry, temporal performances, identification ratio, system design, autonomous system, step response, DABO's Frame, DABO's numbers.**

I. Introduction

In this paper, I present a new way of computing, with precision, for any given stable linear second order dynamical system, whatever its damping ratio and natural pulsation, some temporal performances such as time-response or rise/fall time at any given percentage of the output related to the initial condition or equilibrium. I also present the way to do identification from measurements and also how to design a linear second order dynamical system with precise desired temporal or frequential performances. I, first, give some definitions that I have introduced and present results for an autonomous system, exploring the different cases of critically, overdamped and underdamped systems. Finally, I give briefly some formulas related to a step response, for any linear second order dynamical system.

II. Some definitions

Consider a linear dynamical system of order n given by following equation:

$$a_n \frac{d^n s}{dt^n}(t) + \cdots + a_2 \frac{d^2 s}{dt^2}(t) + a_1 \frac{ds}{dt}(t) + a_0 s(t) = be(t) \quad (1)$$

where all $a_i \neq 0$, for $i \in [0..n]$ and b are all real numbers. Dividing equation (1) by a_0 yields

$$\frac{a_n}{a_0}\frac{d^n s}{dt^n}(t) + \cdots + \frac{a_2}{a_0}\frac{d^2 s}{dt^2}(t) + \frac{a_1}{a_0}\frac{ds}{dt}(t) + s(t) = \frac{b}{a_0}e(t). \quad (2)$$

Independent Researcher, Ziguinchor - Senegal, Africa, e-mail: marcelindaboster@gmail.com.

Then we define following constants for (2):

- $\frac{a_1}{a_0}$ is the line time-constant or the constant of the time-line or the natural time-constant (unit in second)
- $\frac{a_2}{a_0}$ is the surface time-constant or the constant of the surface of time (unit in second square)
- $\frac{a_3}{a_0}$ is the volume time-constant or the constant of the volume of time (unit in second cube)
- $\frac{a_n}{a_0}$ is the constant of the n-dimension space of time (unit in second to the power n).

More generally, any given dynamical system in the sense of differential equations, is the product of a vector of constants of the different dimensions of time times a vector of the quantities whose dynamics we wish to study. It is written as:

$$T.Q = \begin{bmatrix} \frac{a_n}{a_0} & \frac{a_{n-1}}{a_0} & \cdots & \frac{a_2}{a_0} & \frac{a_1}{a_0} \end{bmatrix} \begin{bmatrix} \frac{d^n s}{dt^n} \\ \frac{d^{n-1} s}{dt^{n-1}} \\ \vdots \\ \frac{d^2 s}{dt^2} \\ \frac{ds}{dt} \end{bmatrix} \quad (3)$$

where T is the vector of constants of the different dimensions of time

$$T = \begin{bmatrix} \frac{a_n}{a_0} & \frac{a_{n-1}}{a_0} & \cdots & \frac{a_2}{a_0} & \frac{a_1}{a_0} \end{bmatrix} \quad (4)$$

and Q the vector of the quantities (quantity to study and its derivatives) whose dynamics we wish to study

$$Q = \begin{bmatrix} \frac{d^n s}{dt^n} \\ \frac{d^{n-1} s}{dt^{n-1}} \\ \vdots \\ \frac{d^2 s}{dt^2} \\ \frac{ds}{dt} \end{bmatrix}. \quad (5)$$

It is easy to notice that the left part of equation (2) is exactly

 equal to product $T.Q$ in (3).

III. Second order linear dynamical systems

Consider a second order linear differential equation given by:

$$a_2 \frac{d^2 s}{dt^2}(t) + a_1 \frac{ds}{dt}(t) + a_0 s(t) = be(t) \quad (6)$$

where $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$ and b are all real numbers. Variable t, representing time, is implicit, $e(t)$, is the input and $s(t)$, the output.

Dividing equation (6) by a_0 yields

$$\frac{a_2}{a_0} \frac{d^2 s}{dt^2}(t) + \frac{a_1}{a_0} \frac{ds}{dt}(t) + s(t) = \frac{b}{a_0} e(t). \quad (7)$$

where

- $\tau_n = \frac{a_1}{a_0}$, is the natural time-constant
- $\tau_\tau = \frac{a_2}{a_0}$, is the surface time-constant
- and $G_s = \frac{b}{a_0}$ the static gain.

Written in the classical way with natural pulsation ω_n and damping ratio ξ, both positive and different from zero, yields

$$\frac{d^2 s}{dt^2}(t) + 2\xi\omega_n \frac{ds}{dt}(t) + \omega_n^2 s(t) = b\omega_n^2 e(t) \quad (8)$$

or

$$\frac{1}{\omega_n^2} \frac{d^2 s}{dt^2}(t) + \frac{2\xi}{\omega_n} \frac{ds}{dt}(t) + s(t) = be(t). \quad (9)$$

Therefore, identifying the parameters between equations (6) and (8) yields

$$\begin{cases} a_2 = 1 \\ a_1 = 2\xi\omega_n \\ a_0 = \omega_n^2 \end{cases} \quad (10)$$

which yield

$$\begin{cases} \tau_n = \frac{a_1}{a_0} = \frac{2\xi}{\omega_n} \\ \tau_\tau = \frac{a_2}{a_0} = \frac{1}{\omega_n^2} \end{cases} . \quad (11)$$

Let define the relative time-constant as the ratio between the surface time-constant and the linear time-constant. It is given by:

$$\tau_r = \frac{\tau_\tau}{\tau_n} = \frac{1}{2\xi\omega_n}. \quad (12)$$

Theorem 1. *Let consider a liner second order dynamical system given in (8). There exist, for such a system, only one single and unique time-constant τ_s equal to the double of the relative time-constant τ_r defined in (12):*

$$\tau_s = 2\tau_r = \frac{1}{\xi\omega_n}. \quad (13)$$

A. Autonomous systems

Recall system (6) which characteristic equation is given by following equivalent equations

$$\begin{cases} a_2 r^2 + a_1 r + a_0 = 0 \\ r^2 + 2\xi\omega_n r + \omega_n^2 = 0 \\ \frac{1}{\omega_n^2} r^2 + 2\frac{\xi}{\omega_n} r + 1 = 0 \end{cases} . \quad (14)$$

Let Δ be the discriminant:

$$\Delta = a_1^2 - 4a_2 a_0 = 4\xi^2\omega_n^2 - 4\omega_n^2 = 4\omega_n^2(\xi^2 - 1). \quad (15)$$

Hence, we have the following three distinct cases (with $\omega_n \neq 0$):

- case 1: $\Delta = 0$ or $\xi = 1$
- case 2: $\Delta > 0$ or $\xi > 1$
- case 3: $\Delta < 0$ or $0 < \xi < 1$.

1) Case 1: $\Delta = 0$ ou $\xi = 1$: Therefore the roots of the characteristic polynomial are double and equal to

$$r_1 = r_2 = r = -\frac{a_1}{2a_2} = -\xi\omega_n \quad (16)$$

which yields to solution

$$s(t) = e^{rt} \left(C_{10} t + C_{20} \right) \quad (17)$$

where C_{10} and C_{20} are constants. This means that our system lives on a linear world envelopped with an exponential function.

The double root is represented as it follows

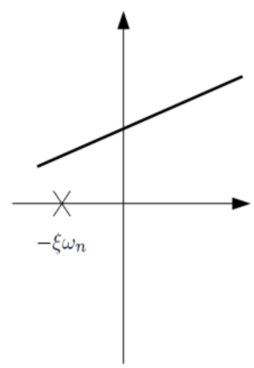

Fig. 1. Roots position $-\xi\omega_n$ on a frame.

For $t = t_D$ we have following relation (t_D is the departure instant)

$$s(t_D) = s_{Dp} = e^{rt_D} \left(C_{10} t_D + C_{20} \right)$$

and

$$C_{10} t_D + C_{20} = s_{Dp} e^{-rt_D}. \quad (18)$$

For the derivative term

$$\frac{ds}{dt}(t) = C_{10} e^{rt} + r(C_{10} t + C_{20}) e^{rt}$$

$$\Leftrightarrow \frac{ds}{dt}(t) = e^{rt} \left[C_{10} + r \left(C_{10} t + C_{20} \right) \right].$$

For $t = t_D$ we then have

$$\frac{ds}{dt}(t_D) = s_{Dv} = e^{rt_D} \left[C_{10} + r(C_{10} t_D + C_{20}) \right]$$

that is equivalent to

$$s_{Dv} e^{-rt_D} = C_{10} + r(C_{10} t_D + C_{20}). \quad (19)$$

However putting (18) in (19) yields

$$s_{Dv} e^{-rt_D} = C_{10} + r(s_{Dp} e^{-rt_D})$$

then

$$C_{10} = e^{-rt_D} \left(s_{Dv} - r s_{Dp} \right).$$

which gives

$$C_{10} = e^{\frac{a_1}{2a_2}t_D}\left(s_{Dv} + \frac{a_1}{2a_2}s_{Dp}\right). \quad (20)$$

From (18) we derive

$$C_{20} = s_{Dp}e^{\frac{a_1}{2a_2}t_D} - t_D e^{\frac{a_1}{2a_2}t_D}\left(s_{Dv} + \frac{a_1}{2a_2}s_{Dp}\right)$$

$$\Leftrightarrow C_{20} = e^{\frac{a_1}{2a_2}t_D}\left(s_{Dp} - t_D s_{Dv} - \frac{a_1}{2a_2}t_D s_{Dp}\right)$$

and

$$C_{20} = e^{\frac{a_1}{2a_2}t_D}\left[s_{Dp}\left(1 - \frac{a_1}{2a_2}t_D\right) - s_{Dv}t_D\right]. \quad (21)$$

Therefore (20) and (21) yield

$$C_{10}t + C_{20} = e^{\frac{a_1}{2a_2}t_D}\left\{t\left(s_{Dv} + \frac{a_1}{2a_2}s_{Dp}\right) + s_{Dp}\left(1 - \frac{a_1}{2a_2}t_D\right) - s_{Dv}t_D\right\}$$

$$\Leftrightarrow C_{10}t + C_{20} = e^{\frac{a_1}{2a_2}t_D}\left\{s_{Dv}(t - t_D) + s_{Dp}\left[1 + \frac{a_1}{2a_2}(t - t_D)\right]\right\}$$

and then for $\xi = 1$ we have

$$s(t) = e^{-\frac{a_1}{2a_2}t}e^{\frac{a_1}{2a_2}t_D}\left\{s_{Dv}(t - t_D) + s_{Dp}\left[1 + \frac{a_1}{2a_2}(t - t_D)\right]\right\}$$

$$\Leftrightarrow s(t) = e^{\frac{a_1}{2a_2}(t_D - t)}\left\{s_{Dv}(t - t_D) + s_{Dp}\left[1 + \frac{a_1}{2a_2}(t - t_D)\right]\right\}$$

$$\Leftrightarrow s(t) = e^{\frac{a_1}{2a_2}(t_D - t)}\left[s_{Dv}(t - t_D) + s_{Dp} + \frac{a_1}{2a_2}s_{Dp}(t - t_D)\right]$$

and finally

$$s(t) = e^{\frac{a_1}{2a_2}(t_D - t)}\left[(t - t_D)\left(s_{Dv} + \frac{a_1}{2a_2}s_{Dp}\right) + s_{Dp}\right] \quad (22)$$

or

$$s(t) = \left[(t - t_D)\left(s_{Dv} + \xi\omega_n s_{Dp}\right) + s_{Dp}\right]e^{-\xi\omega_n(t - t_D)}. \quad (23)$$

For $s_{Dv} = 0$, one has the following expression, that is

$$s(t) = s_{Dp}\left[1 + \xi\omega_n(t - t_D)\right]e^{-\xi\omega_n(t - t_D)}. \quad (24)$$

Let us define $\gamma(t_i)$ a percentage of the response $s(t_i)$ according to the initial condition s_{Dp} at a given instant t_i greater than t_D during transient. Hence, from (24), it is equal to

$$\gamma(t_i) = \frac{s(t_i)}{s_{Dp}} = \left[1 + \xi\omega_n(t_i - t_D)\right]e^{-\xi\omega_n(t_i - t_D)}. \quad (25)$$

Put, for any given instant t_i, such that $t_i > t_D$,

$$t_i - t_D = \alpha\tau_s = \frac{\alpha}{\xi\omega_n} \quad (26)$$

where α is a positive real number different from zero. Putting (26) in (25) yields

$$\gamma(t_i) = \left[1 + \xi\omega_n\frac{\alpha}{\xi\omega_n}\right]e^{-\xi\omega_n\left(\frac{\alpha}{\xi\omega_n}\right)}$$

and finally

$$\gamma(\alpha) = (1 + \alpha)e^{-\alpha}. \quad (27)$$

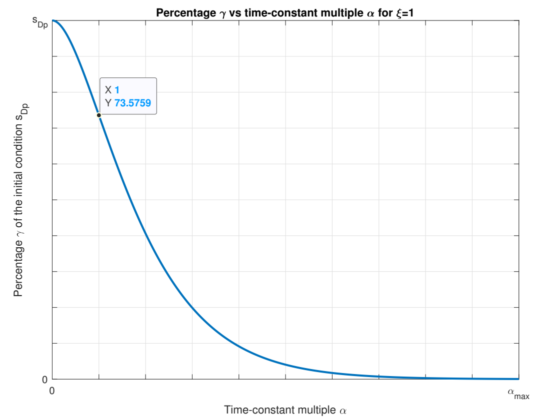

Fig. 2. Percentage $\gamma(\alpha)$ of the initial condition versus multiple α of the time-constant $sdp \neq 0$, with $sdv = 0$, $\xi = 1$.

TABLE I
PERCENTAGE $\gamma_{(\xi=1)}$ OF THE OUTPUT RELATIVE TO THE INITIAL CONDITION s_{Dp} AS A FUNCTION OF THE SYSTEM'S TIME CONSTANT MULTIPLE α (13) (WITH $s_{Dv} = 0$ AND t_D THE DEPARTURE INSTANT).

$\gamma(\alpha)$	95	90	80	74	20	10	5
α	0.356	0.53	0.824	1	3	3.89	4.742

With expression (27), it is impossible to analytically derive an α for a fixed desired percentage value γ. Therefore, numerical resolving yields the following results in Table I, for some percentages. A graphical representation is also given in figure 2.

Note that, for any given percentage γ, during the transient, it is possible to compute its corresponding unique value α.

Remark: for $\alpha = 1$, we have a percentage that is $\gamma = 74\%$. In other words, like for a first-order dynamical system for which the time-constant can be found at $\gamma = 37\%$ (for an autonomous system), for a second-order dynamical system with $\xi = 1$, we have $\gamma = 74\%$ for an instant corresponding to the time-constant defined at (13), for any given value of ω_n.

a) Output time-response at a fixed percentage $\gamma_{\xi=1}$: Put $t_{r_{\gamma_i}}$ the time-response corresponding to percentage γ_i. Put α_i the corresponding coefficient of γ_i, see (27). Hence, the time-response is given by

$$t_{r_{\gamma_i}} = \frac{\alpha_i}{\xi\omega_n} = \frac{\alpha_i}{\omega_n}. \quad (28)$$

because $\xi = 1$.

TABLE II
TIME-RESPONSE $t_{r_{\gamma_i}}$ AT A FIXED PERCENTAGE γ_i FOR $\xi = 1$.

$t_{r_{95}}$	$t_{r_{90}}$	$t_{r_{80}}$	$t_{r_{74}}$	$t_{r_{20}}$	$t_{r_{10}}$	t_{r_5}
$\frac{0.356}{\omega_n}$	$\frac{0.53}{\omega_n}$	$\frac{0.824}{\omega_n}$	$\frac{1}{\omega_n}$	$\frac{3}{\omega_n}$	$\frac{3.89}{\omega_n}$	$\frac{4.742}{\omega_n}$

As an example, consider following dynamical autonomous system given by

$$\frac{d^2s}{dt^2}(t) + 10\frac{ds}{dt}(t) + 25s(t) = 0. \quad (29)$$

By analogy with (8) one has $\xi = 1$ and $\omega_n = 5$. Hence, referring to Table II, the time-response t_{r_5} at 5% of the initial condition of the variable (we suppose the initial condition of the first derivative equal to zero) is given by

$$t_{r_5} = \frac{4.742}{\omega_n} = \frac{4.742}{5} \approx 948 \text{ ms.}$$

By the way,

$$t_{r_{95}} = \frac{0.356}{\omega_n} = \frac{0.356}{5} \approx 71 \text{ ms}$$

and

$$t_{r_{73.58}} = \frac{1}{\omega_n} = \frac{1}{5} = 200 \text{ ms.}$$

We confirm this result with simulations, presented in figure 3.

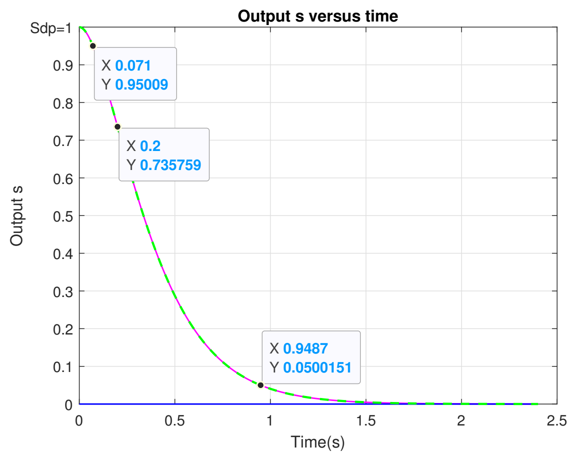

Fig. 3. time-responses for the dynamic of system (29) with $sdp = 1$, $sdv = 0$, $\xi = 1$ and $\omega_n = 5$ rad/s.

b) Output rise/fall time between two distinct percentages (for $\xi = 1$): By the same way, we can also derive from Table I and equation (26) the rise/fall time between two different percentages γ_i and γ_j(27).

TABLE III
RISE/FALL TIME $t_{rf_{\gamma_i/\gamma_j}}$ BETWEEN TWO DISTINCT PERCENTAGES γ_i AND γ_j FOR $\xi = 1$.

$t_{rf_{95/5}}$	$t_{rf_{90/10}}$	$t_{rf_{80/20}}$
$\frac{4.386}{\omega_n}$	$\frac{3.36}{\omega_n}$	$\frac{2.176}{\omega_n}$

As an example, recalling system (29), we have following rise/fall time values

$$t_{rf_{95/5}} = \frac{4.386}{\omega_n} = \frac{4.386}{5} \approx 877 \text{ ms}$$

and

$$t_{rf_{90/10}} = \frac{3.36}{\omega_n} = \frac{3.36}{5} \approx 672 \text{ ms}$$

that are confirmed with the values on figure 4.

c) Identification criterion for a second order dynamical system with $\xi = 1$:

Theorem 2. *Let $R^{2(\xi=1)}_{\gamma_i,\gamma_j}$ be the ratio between two distinct relative instants t_i and t_j such that $t_D < t_i < t_j$. We define it as*

$$R^{2(\xi=1)}_{\gamma_i,\gamma_j} = \frac{t_i - t_D}{t_j - t_D} = \frac{\alpha_i}{\alpha_j} \tag{30}$$

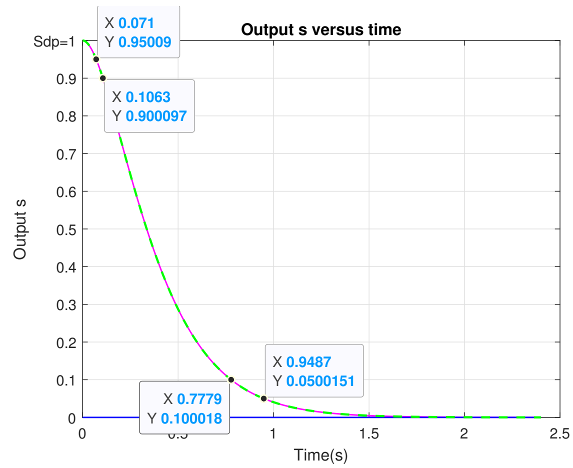

Fig. 4. Fall times for the dynamic of system (29) with $sdp = 1$, $sdv = 0$, $\xi = 1$ and $\omega_n = 5$ rad/s.

where, referring to (26), the relative time-response at a fixed instant is given as

$$t_{i,j} - t_D = \frac{\alpha_{i,j}}{\xi\omega_n}. \tag{31}$$

We then have the following table of ratios that one can use for the identification of a second-order dynamical system with a damping ratio of $\xi = 1$.

TABLE IV
IDENTIFICATION RATIO $R^{2(\xi=1)}_{\gamma_i,\gamma_j}$ BETWEEN TWO DISTINCT PERCENTAGES γ_i AND γ_j DURING TRANSIENT FOR $\xi = 1$.

$R^2_{95,5}$	$R^2_{90,10}$	$R^2_{80,20}$
$\frac{0.356}{4.742} = 0.075$	$\frac{0.53}{3.89} = 0.136$	$\frac{0.824}{3} = 0.275$

Hence, any given autonomous system for which the different ratios of time at the fixed percentages are equal to the values in Table IV is a linear second order system with a damping ratio of 1.

As an example for identification, consider any given response (data with time) of an autonomous second order system with the initial condition $sdp = 1$, $sdv = 0$, $\xi = 1$ and any given natural pulsation. Analyzing those data based on the defined above ratios, yieds to the expected results. And then confirming the damping ratio of $\xi = 1$, one can derive, from any chosen time-response of table II, the system natural pulsation.

d) Design of a second order linear dynamical system with $\xi = 1$: It can be done on the basis of performances of time or those of natural pulsation.

Time-based performances design: our goal is to have a fixed desired time-response or rise/fall time. Going from tables II and III, one can fix the desired temporal performance (with $\xi = 1$) such that

$$t_{des} = \frac{\alpha}{\omega_n}. \tag{32}$$

Therefore ω_n is derived precisely. What yields to a complete dynamical system with $\xi = 1$.

As a first example of design: we want a dynamical system with a damping ratio of $\xi = 1$ (no overshoot on our system) and a time-response at 5% (of the initial condition) of $562\ ms$. Refering to table II, one has

$$t_{r_5} = \frac{4.742}{\omega_n} \Leftrightarrow \omega_n = \frac{4.742}{0.562} \approx 8.438 \text{ rad/s.}$$

Choosing this value implies a fall time between 95% and 5% equal to

$$t_{f_{95/5}} = \frac{4.386}{\omega_n} = \frac{4.386}{8.438} \approx 520 \text{ ms.}$$

One can see, through this example, that it possible to design precisely any desired dynamical system thanks to this new approach I have introduced. See figure 5 for illustration.

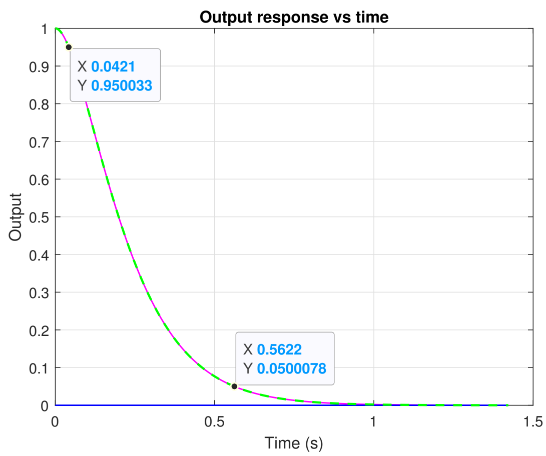

Fig. 5. Desired time-response-based design for a second order dynamical system with $sdp = 1$, $sdv = 0$, $\xi = 1$.

As a second example of design: we want a dynamical (autonomous) second order system with a fall-time that is $t_{r_{80/20}} = 263\ ms$. Then, from table III

$$t_{rf_{80/20}} = \frac{2.176}{\omega_n} \Leftrightarrow \omega_n = \frac{2.176}{0.263} \approx 8.274 \text{ rad/s.}$$

As an illustration of this result, see figure 6.

Natural pulsation-based performance design: with this design, one is able to fix the desired value of natural pulsation of our system with a damping ratio of $\xi = 1$.

As an example, let us fix the desired natural pulsation value equal to $\omega_n = 15$ rad/s. This implies different time-responses that are:

$$t_{r_5} = \frac{4.742}{\omega_n} = \frac{4.742}{15} \Leftrightarrow t_{r_5} \approx 316.1 \text{ ms}$$

$$t_{r_{10}} = \frac{3.89}{\omega_n} = \frac{3.89}{15} \Leftrightarrow t_{r_{10}} \approx 259.3 \text{ ms}$$

$$t_{r_{90}} = \frac{0.53}{\omega_n} = \frac{0.53}{15} \Leftrightarrow t_{r_{90}} \approx 35.3 \text{ ms}$$

and finally, (determining the right α from (27)

$$t_{r_{43.46}} = \frac{1.897}{\omega_n} = \frac{1.897}{15} \Leftrightarrow t_{r_{90}} \approx 126.5 \text{ ms.}$$

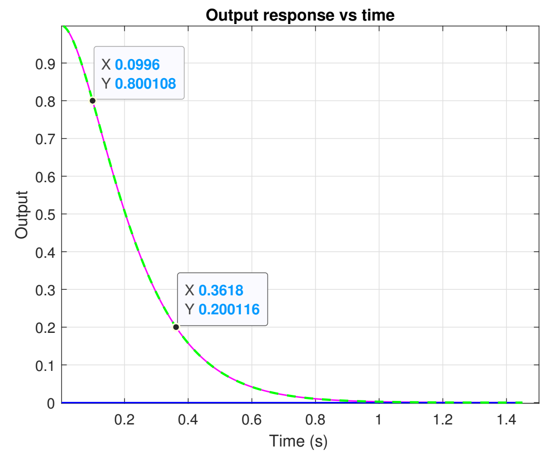

Fig. 6. Desired fall-time-based design for a second order dynamical system with $sdp = 1$, $sdv = 0$, $\xi = 1$.

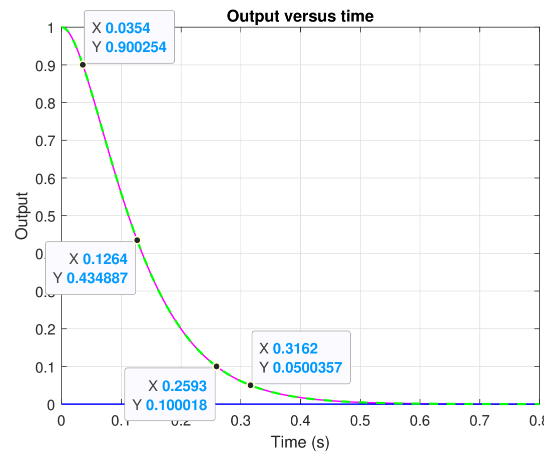

Fig. 7. Impact of natural pulsation-based design on the time-responses with $sdp = 1$, $sdv = 0$, $\xi = 1$.

2) Case 2 : $\Delta > 0$ ou $\xi > 1$: Therefore the roots of the characteristic polynomial are distinct and real

$$\begin{cases} r_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2} \\ r_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_2} \end{cases} \text{or} \quad \begin{cases} r_1 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \\ r_2 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \end{cases}$$

Putting

$$\begin{cases} u = -\xi\omega_n \\ v = \omega_n\sqrt{\xi^2 - 1} \end{cases} \tag{33}$$

yields

$$s(t) = C_{1p}e^{ut}\sinh(vt + C_{2p}) \tag{34}$$

where the C_{ip} are constants with $i = 1, 2$. This means that our system lives in a hyperbolic world envelopped with an exponential function. I choose to present the roots in a new frame that I call DABO's frame with a new family of numbers that I call DABO's numbers.

Definition 3. Let us consider a linear second order dynamical system as presented in (6). I define DABO's numbers (or real conjugated numbers) as any pair of solutions of the characteristic equation of (6) whose discriminant is strictly positive.

Let define $z_1 = a - \gamma b$ and $z_2 = a + \gamma b$, where $\gamma = 1$, a and b two real numbers. Hence we have following properties:

1) Sum: $z_1 + z_2 = a - \gamma b + a + \gamma b = 2a$
2) Product $z_1 z_2 = (a - \gamma b)(a + \gamma b) = a^2 - b^2$
3) Difference $z_1 - z_2 = a - \gamma b - a - \gamma b = 2\gamma b = 2b$
4) Module $\mid z_1 \mid = \mid z_2 \mid = \sqrt{a^2 + b^2}$
5) If $z_1 = a - \gamma b$ is a root of the characteristic equation with real coefficients, then $z_2 = a + \gamma b$ is also a root for the same equation.

This yields, then, to following framework, see figure 8

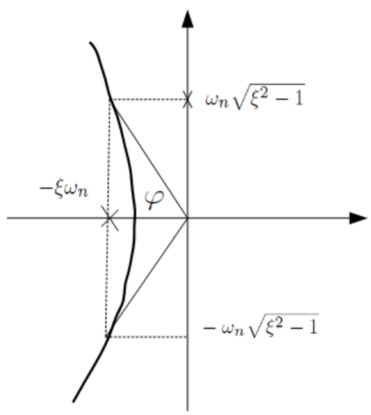

Fig. 8. Roots position on DABO's frame.

For $t = t_D$, one has

$$s(t = t_D) = S_{Dp} = C_{1p} e^{ut_D} \sinh(vt_D + C_{2p})$$

and finally

$$C_{1p} = \frac{S_{Dp} e^{-ut_D}}{\sinh(vt_D + C_{2p})}. \tag{35}$$

The derivative of (34) is given by

$$\frac{d}{dt} s(t) = C_{1p}\left[ue^{ut}\sinh(vt + C_{2p}) + ve^{ut}\cosh(vt + C_{2p})\right]$$

what is equivalent to

$$\frac{d}{dt} s(t) = e^{ut} C_{1p}\left[u\sinh(vt + C_{2p}) + v\cosh(vt + C_{2p})\right]$$

and for $t = t_D$ we have $\frac{d}{dt} s(t = t_D) = s_{Dv}$.

Therefore

$$s_{Dv} = e^{ut_D} C_{1p}\left[u\sinh(vt_D + C_{2p}) + v\cosh(vt_D + C_{2p})\right]. \tag{36}$$

Putting (35) in (36) yields

$$s_{Dv} = us_{Dp} + v\left(e^{ut_D} s_{Dp} e^{-ut_D}\right)\frac{\cosh(vt_D + C_{2p})}{\sinh(vt_D + C_{2p})}$$

and finally

$$C_{2p} = \coth^{-1}\left(\frac{s_{Dv} - us_{Dp}}{vs_{Dp}}\right) - vt_D. \tag{37}$$

Hence, we have

$$s(t) = s_{Dp}\frac{\sinh\left[v(t - t_D) + \coth^{-1}\left(\frac{s_{Dv} - us_{Dp}}{vs_{Dp}}\right)\right]}{\sinh\left[\coth^{-1}\left(\frac{s_{Dv} - us_{Dp}}{vs_{Dp}}\right)\right]} e^{u(t - t_D)} \tag{38}$$

or, putting (33) in (38) yields

$$s(t) = s_{Dp}\frac{\sinh\Big[(t - t_D)\omega_n\sqrt{\xi^2 - 1} + \coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]}{\sinh\Big[\coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]} \times e^{-\xi\omega_n(t - t_D)}. \tag{39}$$

For $s_{Dv} = 0$, equation (39) becomes

$$s(t) = s_{Dp}\frac{e^{-\xi\omega_n(t - t_D)}}{\sqrt{\xi^2 - 1}} \times \sinh\left[\omega_n(t - t_D)\sqrt{\xi^2 - 1} + \coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2 - 1}}\right)\right] \tag{40}$$

because

$$\sinh\left[\coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2 - 1}}\right)\right] = \sqrt{\xi^2 - 1}.$$

Therefore we define, as for (25), a percentage $\gamma(t_i)$ of the response $s(t_i)$ according to the initial condition s_{Dp} at a given moment t_i, greater than t_D, during transient. Hence, it is equal to

$$\gamma(t_i) = \frac{s(t_i)}{s_{Dp}}$$

and then

$$\gamma(t_i) = \frac{\sinh\Big[(t_i - t_D)\omega_n\sqrt{\xi^2 - 1} + \coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]}{\sinh\Big[\coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]} \times e^{-\xi\omega_n(t_i - t_D)}. \tag{41}$$

Using the same paradigm as from relation (26) yields the following

$$\gamma(\alpha, \xi) = \frac{\sinh\Big[\alpha\frac{\sqrt{\xi^2 - 1}}{\xi} + \coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]}{\sinh\Big[\coth^{-1}\Big(\frac{s_{Dv} + \xi\omega_n s_{Dp}}{s_{Dp}\omega_n\sqrt{\xi^2 - 1}}\Big)\Big]} \times e^{-\alpha}. \tag{42}$$

From equation (40) and considering relation (26), one has

$$\gamma(\alpha, \xi) = \frac{e^{-\alpha}}{\sqrt{\xi^2 - 1}} \times \sinh\left[\alpha\frac{\sqrt{\xi^2 - 1}}{\xi} + \coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2 - 1}}\right)\right]. \tag{43}$$

A representation of this percentage is given in figure 9, for some values of the damping ratio: $\xi = 1.2$, $\xi = 1.45$, $\xi = 1.7$ and $\xi = 1.95$.

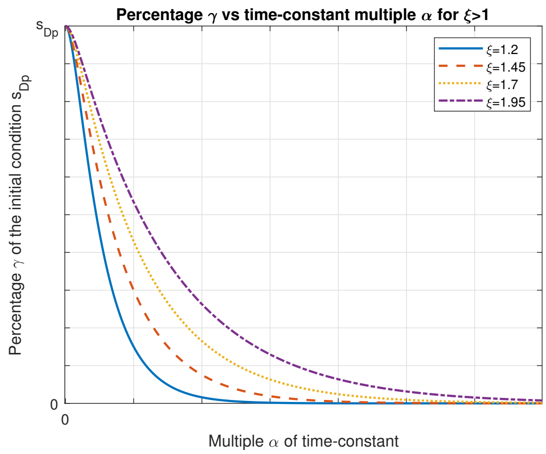

Fig. 9. Percentage $\gamma(\xi, \alpha)$ of the output according to the initial condition versus multiple α of the time-constant ($sdp \neq 0$, with $sdv = 0$, $\xi > 1$).

Hence, searching numerically for the right α for the corresponding fixed γ, we have the following table given for some damping ratio values. But, of course, α can be computed for any given value of the damping ratio $\xi > 1$.

TABLE V
PERCENTAGE $\gamma_{(\xi>1)}$ OF THE OUTPUT RELATIVE TO THE INITIAL CONDITION s_{Dp} AS A FUNCTION OF THE SYSTEM'S TIME CONSTANT MULTIPLE α (13) (WITH $s_{Dv} = 0$ AND t_D AN ARBITRARY DEPARTURE INSTANT).

$\gamma(\alpha)$	95	90	80	20	10	5
$\alpha_{\xi=1.1}$	0.4	0.6	0.9	3.6	4.8	6
$\alpha_{\xi=1.25}$	0.5	0.7	1.1	4.7	6.5	8.2
$\alpha_{\xi=1.5}$	0.6	0.9	1.4	6.9	9.7	12.3
$\alpha_{\xi=1.75}$	0.7	1.1	1.8	9.5	13.4	17.3
$\alpha_{\xi=1.9}$	0.8	1.2	2	11.3	15.9	20.6

Following table presents, for some damping ratios, the percentage at which one can find the exact value of the second-order time-constant as defined in (13).

TABLE VI
SOME VALUES OF PERCENTAGE $\gamma_{(\xi>1)}$ AT A RELATIVE MOMENT ($t_i - t_D = \frac{1}{\xi\omega_n}$) THAT IS EQUAL TO THE DEFINED SYSTEM'S TIME-CONSTANT (13) FOR SOME DAMPING RATIO VALUES (WITH $s_{Dv} = 0$ AND t_D AN ARBITRARY DEPARTURE MOMENT).

ξ	1.1	1.25	1.5	1.75	1.9
$\gamma(\%)$	78	83	88	91	92.3

More generally speaking, just put $\alpha = 1$ for any given $\xi > 1$ in equation (43) and finally one have

$$\gamma(\xi > 1)_{\alpha=1} = 0.37 \frac{\sinh\left[\frac{\sqrt{\xi^2-1}}{\xi} + \coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2-1}}\right)\right]}{\sqrt{\xi^2-1}}. \tag{44}$$

a) Output time-response at a fixed percentage $\gamma_{\xi>1}$: It is defined as done for formula (28).

As an example, let us consider following system

$$\frac{d^2 s}{dt^2}(t) + 15\frac{ds}{dt}(t) + 15s(t) = 0. \tag{45}$$

TABLE VII
NATURAL PULSATION TIMES TIME-RESPONSE AT A FIXED PERCENTAGE γ VERSUS DAMPING RATIO.

ξ	1.1	1.25	1.5	1.75	1.9
$\omega_n t_{r_{100/2}}$	6.9	8.4	10.7	12.8	14.5
$\omega_n t_{r_{100/5}}$	5.5	6.6	8.2	9.9	10.8
$\omega_n t_{r_{100/10}}$	4.4	5.2	6.4	7.7	8.4
$\omega_n t_{r_{100/20}}$	3.3	3.8	4.6	5.5	6.0

We derive, by identification of equation parameters, that

$$\omega_n = \sqrt{15} \approx 3.873 \text{ rad/s}$$

and

$$\xi = \frac{15}{2 * \sqrt{15}} \approx 1.9365.$$

Let us compute time-responses at different chosen percentages of the output related to the initial condition.

- time-response at 5% is given by
$$t_{r_5} = \frac{\alpha_5}{\xi\omega_n} = \frac{21.415}{1.9365 * 3.873} \approx 2.855 \text{ s.}$$
- time-response at 39.33% is given by
$$t_{r_{39.33}} = \frac{\alpha_{39.33}}{\xi\omega_n} = \frac{7.057}{1.9365 * 3.873} \approx 941 \text{ ms.}$$

where $\alpha_5 = 21.415$ and $\alpha_{39.33} = 7.057$ are both computed, numerically, from (43). The computed time-responses are then confirmed by simulation. Refer to figure 10.

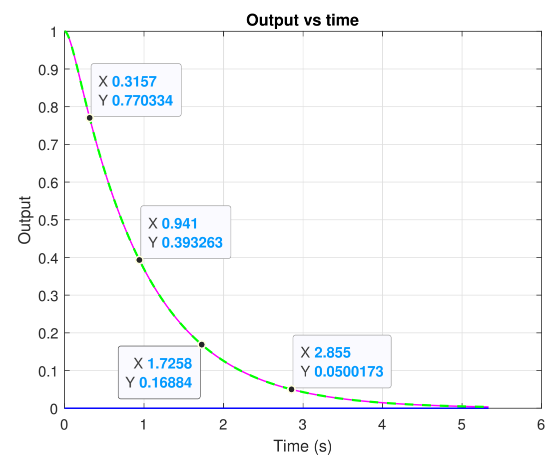

Fig. 10. time-responses for the dynamic of system (45) with $sdp = 1$, $sdv = 0$, $\xi = 1.9365$ and $\omega_n = 3.873$ rad/s.

b) Output rise/fall time between two distinct percentages (for $\xi > 1$) : For two distinct fixed percentages γ_i and γ_j the rise/fall time is computed as following:

$$t_{r/f_{\gamma_i\gamma_j}} = \frac{\alpha_j}{\xi\omega_n} - \frac{\alpha_i}{\xi\omega_n} = \frac{\alpha_j - \alpha_i}{\xi\omega_n} \tag{46}$$

which yields to following results for some fixed percentages

Recall system (45). Its fall time between 77.03% and 16.88% can be computed as following:

$$t_{f_{77.03/16.88}} = \frac{\alpha_{16.88} - \alpha_{77.03}}{\xi\omega_n} = \frac{12.945 - 2.368}{1.9365 * 3.873} \approx 1.41 \text{ s.}$$

This result is confirmed in figure 10.

TABLE VIII
NATURAL PULSATION TIMES RISE/FALL TIME BETWEEN TWO DISTINCT FIXED PERCENTAGES $\gamma_{i,j}$ VERSUS DAMPING RATIO.

ξ	1.1	1.25	1.5	1.75	1.9
$\omega_n t_{rf98/2}$	6.7	8.2	10.4	12.6	13.8
$\omega_n t_{rf95/5}$	5.1	6.2	7.8	9.5	10.4
$\omega_n t_{rf90/10}$	3.9	4.6	5.9	7.1	7.8
$\omega_n t_{rf80/20}$	2.5	2.9	3.7	4.4	4.9

c) Identification criterion for a second order linear dynamical system with $\xi > 1$: It is done following the same principle as the identification for a damping ratio $\xi = 1$ presented previously. Hence, in table IX are resumed the different values of the identification ratio at different percentages and for some damping ratio.

TABLE IX
IDENTIFICATION RATIO VALUES FOR SOME DAMPING RATIO VALUES.

ξ	1.1	1.25	1.5	1.75	1.9
$R^2_{98,2}$	0.03	0.026	0.021	0.0178	0.0163
$R^2_{95,5}$	0.065	0.055	0.046	0.039	0.036
$R^2_{90,10}$	0.123	0.106	0.09	0.079	0.074
$R^2_{80,20}$	0.256	0.233	0.206	0.187	0.18

d) Design of a second order linear dynamical system with $\xi > 1$: It can be done on the basis of performances of time or on those of natural pulsation.

Time-based performances design:

Recall the given example for the case $\xi = 1$. Our fixed goal is to reach a time-response at 5% of the initial condition $sdp = 1$ that is equal to 562 ms. Let us put, for this example, $\xi = 1.75$ (an overdamped system). We then have following natural pulsation, refer to table VII:

$$t_{f_{100/5}} = \frac{9.9}{\omega_n} \Leftrightarrow \omega_n = \frac{9.9}{0.562} \approx 17.6157 \text{ rad/s.}$$

Plotting system response versus time for both pairs of damping ratio and natural pulsation ($\xi = 1$, $\omega_n = 8.438$ rad/s and $\xi = 1.75$, $\omega_n = 17.6157$ rad/s) confirms that we get our desired time-response that is $t_{r_5} = t_{f_{100/5}} = 562\ ms$. See figure 11.

We can see that, whatever the damping ratio, it is possible to reach a fixed time-response value using our appropriate precise formulas for design.

Natural pulsation-based performance design: fixing a desired value for natural pulsation together with a damping ratio yields to specific time-responses, as seen in the previous case.

As an example, let us fix the desired natural pulsation value equal to $\omega_n = 15$ rad/s and a damping ratio of $\xi = 1.632$. This implies different time-responses that can be computed as it follows:

$$t_{r_5} = \frac{\alpha_5}{\xi\omega_n} = \frac{14.878}{1.632\omega_n} = \frac{9.12}{\omega_n} \Leftrightarrow t_{r_5} \approx 608 \text{ ms}$$

and

$$t_{r_{42}} = \frac{\alpha_{42}}{\xi\omega_n} = \frac{4.73}{1.632\omega_n} = \frac{2.9}{\omega_n} \Leftrightarrow t_{r_{42}} \approx 193 \text{ ms}$$

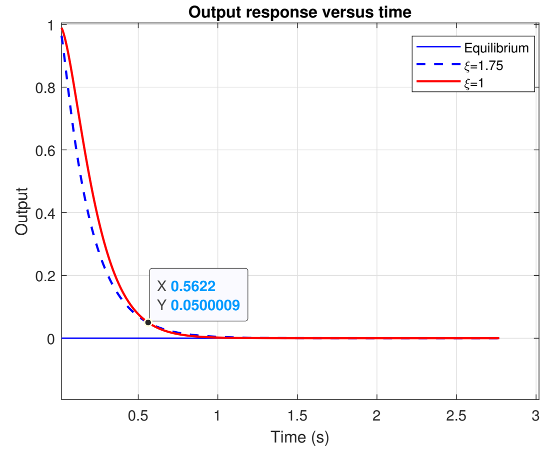

Fig. 11. Desired time-response-based design for two second order dynamical systems with $sdp = 1$, $sdv = 0$, $\xi = 1$ (continuous line) and $\xi = 1.75$ (dashed line).

with α_5 and α_{42} computed, numerically, from equation (43) with $\xi = 1.632$. See figure 12 for an illustration.

The choosen percentages shown here is to demonstrate to the reader that, it is possible, with our innovation, to compute precisely time-responses whatever the point on the response curve in the transient.

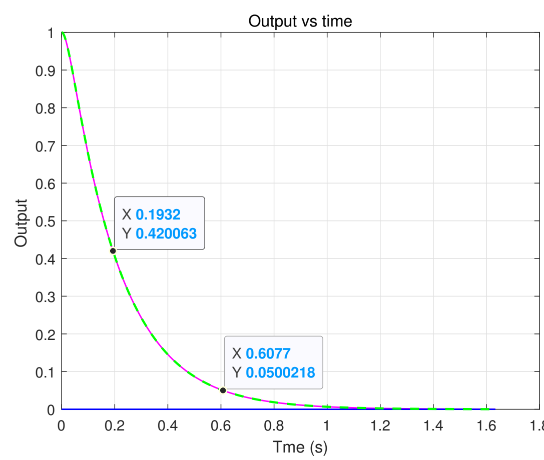

Fig. 12. Impact of natural pulsation-based design on the time-responses with $sdp = 1$, $sdv = 0$, $\xi = 1.632$.

3) Case 3 : $\Delta < 0$ ou $\xi < 1$: Therefore the roots of the characteristic polynomial are distinct and complex conjugate

$$\begin{cases} r_1 = \frac{-a_1 - i\sqrt{|\Delta|}}{2a_2} \\ r_2 = \frac{-a_1 + i\sqrt{|\Delta|}}{2a_2} \end{cases} \text{ou} \begin{cases} r_1 = -\xi\omega_n - i\omega_n\sqrt{1-\xi^2} \\ r_2 = -\xi\omega_n + i\omega_n\sqrt{1-\xi^2} \end{cases}$$

and the solution of the autonomous system is given by

$$s(t) = e^{ut} C_{1n} \sin(vt + C_{2n}) \tag{47}$$

where the C_{in} are constants with $i = 1, 2$ and $u = -\xi\omega_n$ and $v = \omega_n\sqrt{1-\xi^2}$. What yields

$$-\frac{u}{v} = \frac{\xi}{\sqrt{1-\xi^2}} \tag{48}$$

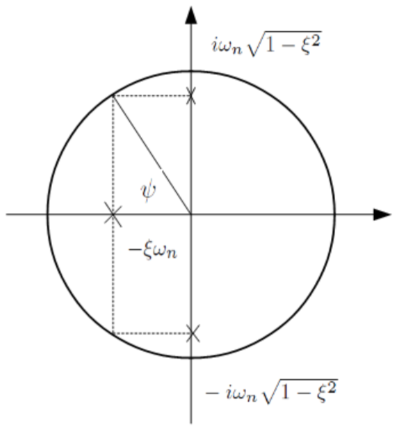

Fig. 13. Roots position on the complex frame.

For $t = t_D$, one has

$$s(t_D) = s_{Dp} = e^{ut_D} C_{1n} sin\left(vt_D + C_{2n}\right)$$

$$C_{1n} = \frac{s_{Dp} e^{-ut_D}}{\sin\left(vt_D + C_{2n}\right)}. \tag{49}$$

The derivative of the output is equal to:

$$\frac{ds}{dt}(t) = C_{1n}\left[ue^{ut}\sin\left(vt + C_{2n}\right) + ve^{ut}\cos\left(vt + C_{2n}\right)\right]$$

$$\Leftrightarrow \frac{ds}{dt}(t) = ue^{ut} C_{1n}\sin\left(vt + C_{2n}\right) + ve^{ut} C_{1n}\cos\left(vt + C_{2n}\right).$$

For $t = t_D$, one has:

$$\frac{ds}{dt}(t_D) = ue^{ut_D} C_{1n}\sin\left(vt_D + C_{2n}\right) + ve^{ut_D} C_{1n}\cos\left(vt_D + C_{2n}\right) = s_{Dv}.$$

From (49), we have

$$s_{Dp} e^{-ut_D} = C_{1n}\sin\left(vt_D + C_{2n}\right)$$

and consequently

$$s_{Dv} = ue^{ut_D} s_{Dp} e^{-ut_D} + ve^{ut_D}\frac{s_{Dp} e^{-ut_D}}{\sin\left(vt_D + C_{2n}\right)}\cos\left(vt_D + C_{2n}\right)$$

$$\Leftrightarrow s_{Dv} = us_{Dp} + ve^{ut_D} s_{Dp} e^{-ut_D}\cot\left(vt_D + C_{2n}\right)$$

$$\Leftrightarrow \frac{s_{Dv} - us_{Dp}}{vs_{Dp}} = \cot\left(vt_D + C_{2n}\right)$$

$$vt_D + C_{2n} = \cot^{-1}\left(\frac{s_{Dv} - s_{Dp}u}{s_{Dp}v}\right)$$

hence

$$C_{2n} = \cot^{-1}\left(\frac{s_{Dv} - s_{Dp}u}{s_{Dp}v}\right) - vt_D. \tag{50}$$

Finaly, for

$$s(t) = \frac{\sin\left(vt + C_{2n}\right)}{\sin\left(vt_D + C_{2n}\right)} e^{ut} e^{-ut_D} s_{Dp}$$

that gives, with equation (50),

$$s(t) = s_{Dp}\frac{\sin\left[v\left(t - t_D\right) + \cot^{-1}\left(\frac{s_{Dv} - s_{Dp}u}{vs_{Dp}}\right)\right]}{\sin\left[\cot^{-1}\left(\frac{s_{Dv} - s_{Dp}u}{vs_{Dp}}\right)\right]} e^{u(t - t_D)} \tag{51}$$

or

$$s(t) = s_{Dp}\frac{\sin\left[\omega_n\left(t - t_D\right)\sqrt{1-\xi^2} + \cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]}{\sin\left[\cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]} \times e^{-\xi\omega_n(t - t_D)}. \tag{52}$$

what gives, for $S_{Dv} = 0$ and $S_{Dp} \neq 0$ that

$$s(t) = s_{Dp}\frac{\sin\left[\omega_n\left(t - t_D\right)\sqrt{1-\xi^2} + \cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]}{\sin\left[\cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]} \times e^{-\xi\omega_n(t - t_D)}. \tag{53}$$

or

$$s(t) = s_{Dp}\frac{e^{-\xi\omega_n(t - t_D)}}{\sqrt{1-\xi^2}} \times \sin\left[\omega_n\left(t - t_D\right)\sqrt{1-\xi^2} + \cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]. \tag{54}$$

because

$$\sin\left[\cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right] = \sqrt{1-\xi^2}.$$

Therefore we define, as for (25), a percentage $\gamma(t_i)$ that it is equal to

$$\gamma(t_i) = \frac{\sin\left[\left(t - t_D\right)\omega_n\sqrt{1-\xi^2} + \cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]}{\sin\left[\cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]} \times e^{-\xi\omega_n(t_i - t_D)}. \tag{55}$$

Using the same paradigm as from relation (26) yields the following

$$\gamma(\alpha, \xi) = \frac{\sin\left[\alpha\frac{\sqrt{1-\xi^2}}{\xi} + \cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]}{\sin\left[\cot^{-1}\left(\frac{s_{Dv} + s_{Dp}\xi\omega_n}{s_{Dp}\omega_n\sqrt{1-\xi^2}}\right)\right]} \times e^{-\alpha}. \tag{56}$$

From equation (54) and considering relation (26), one has

$$\gamma(\alpha,\xi) = \frac{e^{-\alpha}}{\sqrt{1-\xi^2}} \times \sin\left[\alpha\frac{\sqrt{1-\xi^2}}{\xi} + \cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]. \quad (57)$$

A graphical representation of (57) is given in figure 14, for some values of the damping ratio: $\xi = 0.2$, $\xi = 0.45$, $\xi = 0.7$ and $\xi = 0.85$.

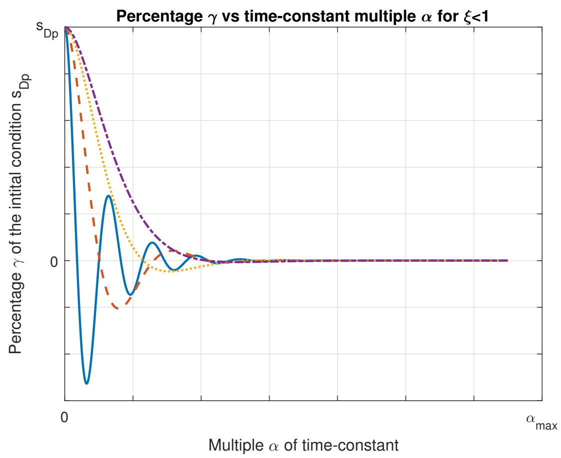

Fig. 14. Percentage $\gamma(\xi,\alpha)$ versus multiple α of the time-constant $sdp \neq 0$, with $sdv = 0$, $\xi < 1$.

Please note that, from relation (57), it is possible to compute, numerically, all values of α for any given percentage $\gamma_{\xi<1}$ during transient.

TABLE X
PERCENTAGE $\gamma_{(\xi<1)}$ AS A FUNCTION OF THE SYSTEM'S TIME-CONSTANT MULTIPLE α (13) (WITH $s_{Dv} = 0$ AND t_D AN ARBITRARY DEPARTURE INSTANT).

$\gamma(\alpha)$	95	90	80	20	10	5
$\alpha_{\xi=0.05}$	0.016	0.023	0.032	0.07	0.076	0.078
$\alpha_{\xi=0.35}$	0.115	0.166	0.244	0.591	0.652	0.685
$\alpha_{\xi=0.7}$	0.24	0.353	0.532	1.563	1.841	2.029
$\alpha_{\xi=0.85}$	0.297	0.44	0.672	2.19	2.706	3.13
$\alpha_{\xi=0.97}$	0.343	0.513	0.793	2.818	3.627	4.384

Table XI presents, for some damping ratio values, the percentage at which one can find the exact value of the second-order time-constant as defined in (13).

TABLE XI
SOME VALUES OF PERCENTAGE $\gamma_{(\xi<1)}$ AT A RELATIVE MOMENT $(t_i - t_D = \frac{1}{\xi\omega_n})$ THAT IS EQUAL TO THE DEFINED SYSTEM'S TIME-CONSTANT (13) FOR SOME DAMPING RATIO VALUES (WITH $s_{Dv} = 0$ AND t_D AN ARBITRARY DEPARTURE MOMENT).

ξ	0.05	0.35	0.7	0.85	0.97
$\gamma(\%)$	17.5	-27	50	64.4	72

In general, just put $\alpha = 1$ for any given $\xi < 1$ in equation (57) and finally

$$\gamma(\xi<1)_{\alpha=1} \approx 0.37\frac{\sin\left[\frac{\sqrt{1-\xi^2}}{\xi} + \cot^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]}{\sqrt{1-\xi^2}}. \quad (58)$$

a) Output time-response at a fixed percentage $\gamma_{\xi<1}$: It is given by the expressed formula in equation (28). Hereunder is a table of time-response versus damping ratio and natural pulsation.

TABLE XII
NATURAL PULSATION TIMES TIME-RESPONSE AT A FIXED PERCENTAGE VERSUS DAMPING RATIO.

ξ	0.1	0.5	0.7	0.85	0.95
$Undershoot1\%$	-72.92	-16.30	-4.6	-0.63	-0.01
$\omega_n t_{rf_{100/2}}$	1.66	2.354	3.109	4.188	5.263
$\omega_n t_{rf_{100/5}}$	1.62	2.264	2.9	3.682	4.372
$\omega_n t_{rf_{100/10}}$	1.56	2.126	2.629	3.185	3.642
$\omega_n t_{rf_{100/20}}$	1.45	1.886	2.233	2.578	2.851

Consider following dynamical system:

$$\frac{d^2s}{dt^2}(t) + 5.42\frac{ds}{dt}(t) + 15s(t) = 0. \quad (59)$$

By identification, one has

$$\omega_n = \sqrt{15} \approx 3.873 \text{ rad/s}$$

and

$$\xi = \frac{15}{2*\sqrt{15}} \approx 0.7.$$

From table XII, one has the system time-response at 5% that is

$$t_{r_5} = \frac{2.9}{\omega_n} = \frac{2.9}{3.873} \approx 749 \text{ ms}.$$

And, deriving numerically, the right value for α from (57) the system time-response at 63% is given by

$$t_{r_{63}} = \frac{1.139}{\omega_n} = \frac{1.139}{3.873} \approx 294 \text{ ms}.$$

The computed time-responses are then confirmed by simulation. Refer to figure 15.

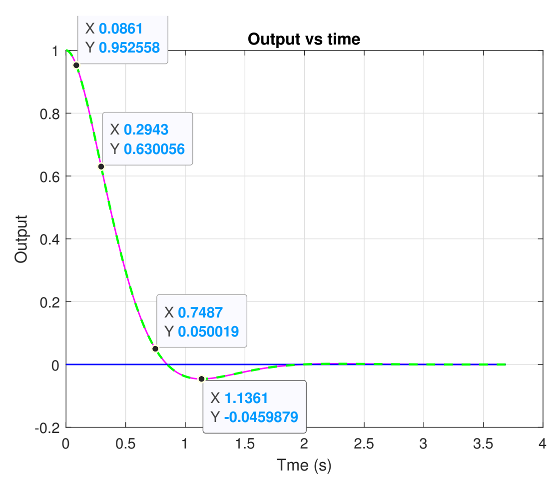

Fig. 15. Time-responses for the dynamic of system (45) with $sdp = 1$, $sdv = 0$, $\xi = 0.7$ and $\omega_n = 3.873$ rad/s.

b) Output rise/fall time between two distinct percentages for $\xi < 1$: For two distinct fixed percentages γ_i and γ_j the rise/fall time is computed as following:

$$t_{r/f_{\gamma_i \gamma_j}} = \frac{\alpha_j}{\xi \omega_n} - \frac{\alpha_i}{\xi \omega_n} = \frac{\alpha_j - \alpha_i}{\xi \omega_n} \quad (60)$$

which yields to results in table $XIII$.

TABLE XIII
NATURAL PULSATION TIMES TIME-RESPONSE AT A FIXED PERCENTAGE VERSUS DAMPING RATIO.

ξ	0.1	0.5	0.7	0.85	0.95
$Undershoot1\%$	-72.92%	-16.30	-4.6	-0.63	-0.01
$\omega_n t_{rf98/2}$	1.46	2.148	2.889	3.975	5.05
$\omega_n t_{rf95/5}$	1.3	1.928	2.557	3.334	4.018
$\omega_n t_{rf90/10}$	1.1	1.638	2.124	2.668	3.114
$\omega_n t_{rf80/20}$	0.79	1.162	1.473	1.787	2.037

Recall system (59). Its fall time between 95% and 5% can be computed as following:

$$t_{f_{95/5}} = \frac{2.557}{\omega_n} = \frac{2.557}{3.873} \approx 660.2 \text{ ms.}$$

This result is confirmed with data tips in figure 15.

c) Identification criterion for a second order linear dynamical system with $\xi < 1$: As for the previous cases, refer to table XIV for some damping ratios.

TABLE XIV
IDENTIFICATION RATIO VALUES FOR SOME DAMPING RATIOS $\xi < 1$.

ξ	0.1	0.5	0.7	0.85	0.95
$R^2_{98,2}$	0.121	0.088	0.068	0.0178	0.041
$R^2_{95,5}$	0.198	0.148	0.118	0.039	0.081
$R^2_{90,10}$	0.293	0.23	0.192	0.079	0.145
$R^2_{80,20}$	0.455	0.383	0.34	0.187	0.285

d) Design of a second order linear dynamical system with $\xi < 1$: It can be done on the basis of performances of time or on that of natural pulsation.

Time-based performances design:

Recall previous example for which $\xi = 0.7$ (59). Our fixed goal is to reach a time-response at 5% of the initial condition $sdp = 1$ that is equal to 562 ms. We then have following natural pulsation, refer to table XII

$$t_{f_{100/5}} = \frac{2.9}{\omega_n} \Leftrightarrow \omega_n = \frac{2.9}{0.562} \approx 5.16 \text{ rad/s.}$$

One can notice that, despite of different values of the damping ratio, we have exactly the same desired time-response. This result is possible thanks to my precise design formulas. Simulations are done with following pairs: $\xi = 0.7$, $\omega_n =$ 5.16 rad/s, $\xi = 1$, $\omega_n = 8.438$ rad/s and $\xi = 1.75$, $\omega_n =$ 17.6157 rad/s.

Of course, this result can be generalised for any desired time-response with any desired profile of response (critically damped, underdamped and overdamped systems) for the studied system.

Natural pulsation-based performance design: As the two previous examples, let us fix the desired natural pulsation value

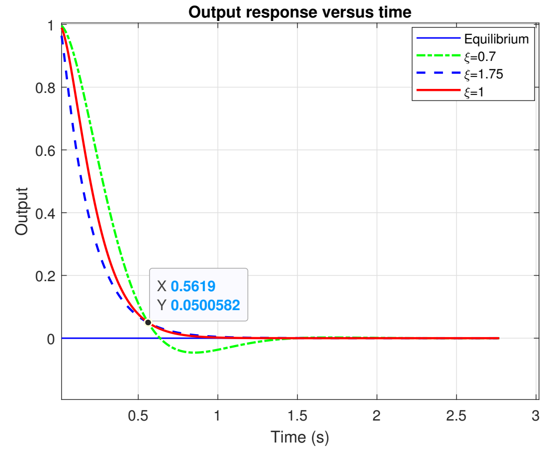

Fig. 16. Desired time-response-based design for three second order dynamical systems with $sdp = 1$, $sdv = 0$, $\xi = 0.7$ (dash-dot line), $\xi = 1$ (continuous line) and $\xi = 1.75$ (dashed line).

equal to $\omega_n = 15$ rad/s and a damping ratio of $\xi = 0.437$. This implies different time-responses that can be computed as it follows:

$$t_{r_7} = \frac{\alpha_7}{\xi \omega_n} = \frac{0.906}{0.437 * 15} \Leftrightarrow t_{r_7} \approx 138.2 \text{ ms}$$

and

$$t_{r_{83}} = \frac{\alpha_{83}}{\xi \omega_n} = \frac{4.73}{0.437 * 15} \Leftrightarrow t_{r_{83}} \approx 43.2 \text{ ms}$$

where α_7 and α_{83} are both computed, numerically, from equation (57) with $\xi = 0.437$. See figure 17 for an illustration.

The choosen percentages shown here are to demonstrate to the reader that, it is possible, with our innovation, to compute precisely time-responses whatever the point on the response curve in the transient.

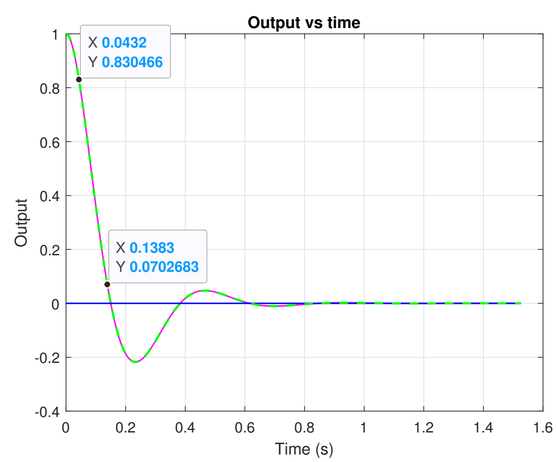

Fig. 17. Impact of natural pulsation-based design on the time-responses with $sdp = 1$, $sdv = 0$, $\xi = 0.437$ and $\omega_n = 15$ rad/s.

4) Summary for autonomous systems: Please notice that the autonomous system responses are all enveloped by the same exponential function and multiplied by:

- an affine function of time for a critically damped system ($\xi = 1$)

$$s\left(t\right) = s_{Dp}\left[1 + \xi\omega_n\left(t - t_D\right)\right]e^{-\xi\omega_n\left(t - t_D\right)} \tag{61}$$

- a hyperbolic function of time for an overdamped system ($\xi > 1$)

$$s(t) = s_{Dp}\frac{e^{-\xi\omega_n(t-t_D)}}{\sqrt{\xi^2 - 1}} \times \sinh\left[\omega_n\left(t - t_D\right)\sqrt{\xi^2 - 1} + \coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2 - 1}}\right)\right] \tag{62}$$

- and a circular function of time for an underdamped system ($\xi < 1$)

$$s(t) = s_{Dp}\frac{e^{-\xi\omega_n(t-t_D)}}{\sqrt{1 - \xi^2}} \times \sin\left[\omega_n\left(t - t_D\right)\sqrt{1 - \xi^2} + \cot^{-1}\left(\frac{\xi}{\sqrt{1 - \xi^2}}\right)\right]. \tag{63}$$

with $s_{Dp} \neq 0$ and $s_{Dv} = 0$.

We can say that a linear second order dynamical system, according to its damping ratio, lives in three different worlds: the affine world, the circular world and the hyperbolic world.

B. Step responses

In the case of an input signal $e(t) = K_0$ for our system (7),

$$\frac{a_2}{a_0}\frac{d^2s}{dt^2}\left(t\right) + \frac{a_1}{a_0}\frac{ds}{dt}\left(t\right) + s\left(t\right) = \frac{bK_0}{a_0} \tag{64}$$

we have following responses for the different cases related to the damping ratio with all initial conditions of the variable and its derivative equal to zero ($s_{Dp} = s_{Dv} = 0$).

- For a critically damped system ($\xi = 1$)

$$s(t) = \frac{bK_0}{a_0}\left[1 - \left[1 + \xi\omega_n\left(t - t_D\right)\right]e^{-\xi\omega_n\left(t - t_D\right)}\right]. \tag{65}$$

- For an overdamped system ($\xi > 1$)

$$s\left(t\right) = \frac{bK_0}{a_0}\left(1 - \frac{e^{-\xi\omega_n\left(t-t_D\right)}}{\sqrt{\xi^2 - 1}}\sinh\left[\omega_n\left(t - t_D\right)\sqrt{\xi^2 - 1} + \coth^{-1}\left(\frac{\xi}{\sqrt{\xi^2 - 1}}\right)\right]\right). \tag{66}$$

- And for an underdamped system ($\xi < 1$)

$$s(t) = \frac{bK_0}{a_0}\left(1 - \frac{e^{-\xi\omega_n\left(t-t_D\right)}}{\sqrt{1 - \xi^2}}\sin\left[\omega_n\left(t - t_D\right)\sqrt{1 - \xi^2} + \cot^{-1}\left(\frac{\xi}{\sqrt{1 - \xi^2}}\right)\right]\right). \tag{67}$$

It is easy to notice that the equilibrium is equal to $Eq = \frac{bK_0}{a_0}$. Please note that all fixed percentages, for which different temporal performances have been evaluated for autonomous systems must be exactly the complement at 100% in the case of step responses.

IV. Conclusions

In this paper I have shown that it is possible to compute, precisely, temporal performances for any given stable second order linear dynamical system whatever its damping ratio and natural pulsation parameters. This is possible because I have introduced a new important result: there is only one single and unique time-constant for any second order ordinary differential equation. It is also possible because of a new paradigm that is any linear second order dynamical system, while evolving, makes steps in time that are multiples of its own time-constant as I define it for second order ODEs.

References

[1] - M. N. DABO (2024) - SYSTEMES DYNAMIQUES: nouvelle vision - nouveau paradigme - nouvelles méthodes pour la caractérisation, l'identification, le design et la commande.

French version book. Order here. English coming soon !

Youtube: Please Like Subscribe Share Support.Videos

Acknowlegment

I dedicate this work to my parents, in humble recognition that "honor your father and your mother" is the first commandment with promise (Ephesians 6:2-3), and in gratitude for the foundation they have provided for my academic journey and life.

"*I give praise to you, Father, Lord of heaven and earth, for although you have hidden these things from the wise and the learned you have revealed them to the childlike. Yes, Father, such has been your gracious will.*" Matthew 11, 25 - 26.

To the only Glory of GOD The Father, JESUS-CHRIST our Lord, his Son and the HOLY-SPIRIT. Amen.

Wishing you happiness, blessings, graces and success in all your projects and endeavors.

Dr. Marcelin Nini DABO is a Kamit from SUNUGAAL (African from SENEGAL) . He is an independent researcher in Sciences and Techniques and a passionate for African culture. Dr. DABO earns his PhD, Master / Engineer and Technician degrees in 2010, 2006 and 2002, respectively. He is author of "SYSTEMES DYNAMIQUES: nouvelle vision - nouveau paradigme - nouvelles méthodes" a book in which he gives much more details related to this paper.